Advanced Numerical Techniques for Financial Engineering

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Abstract

We present some aspects of advanced numerical analysis for the pricing and risk managment of financial derivativess within a partial differential equation framework.

Introduction

The Pricing of structured financial instruments is a major issue in modern risk management, and the more complicated the instruments become, the more it is important to use advanced numerical pricing schemes. In this article, we concentrate on schemes for financial models which lead to partial differential equations. This covers a wide range of models, like Black-Scholes, certain models of stochastic volatility and the wide range of (one or more factor) short rate models like Hull-White and Black-Karasinski.

Green´s Functions and Adaptive Integration

Let us start with the most easy case of classical Black-Scholes: Assume, the underlying equity follows a random walk

$$dS(t) = \mu S(t)\, dt + \sigma S\, dW$$

with dW being the increment of a standard Wiener process. The Black-Scholes trick constructs a risk-free portfolio of being short one option and long *delta* shares which leads to an evolution of the portfolio value which is risk-free and therefore has to lead to the same return as the risk-free cash account. The value of a European option within this framework satisfies the Black Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - rV = 0$$

This is a parabolic differential equation backwards in time. To make it well-posed (meaning uniquely solvable with the solution depending continuous on the data), one has to specify an end condititon (the pay-off function at maturity) and boundary conditions at zero and at infinity.

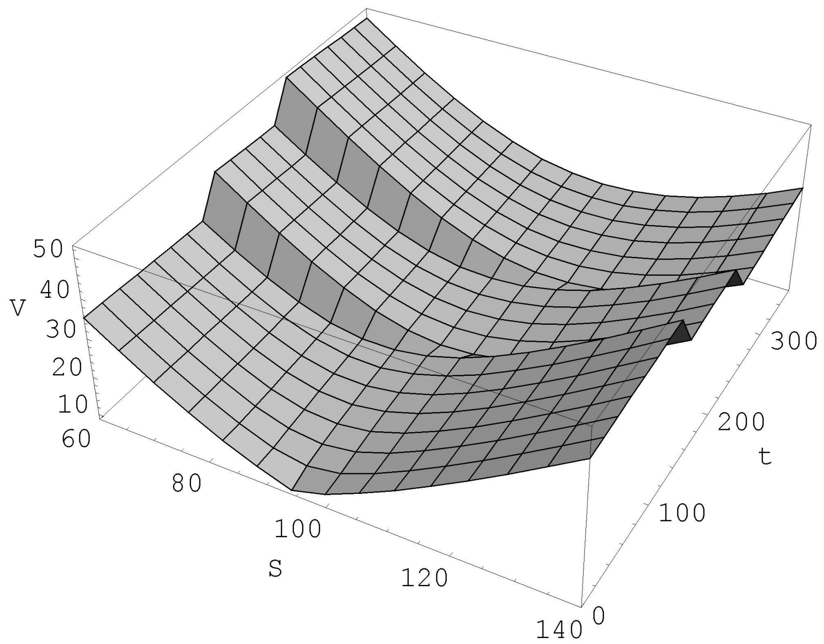
The solution of this problem is given by

$$V(S, t) = \int_0^\infty G(S, S1, t, T)\, \mathrm{Payoff}(S1)\, dS1$$

with

$$G(S, S1, t, T) = \frac{e^{-r(T-t)}}{\sigma S1 \sqrt{2\pi(T-t)}}\, e^{\frac{-\left(\log(S/S1) + \left(r - (1/2)\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}}$$

for arbitrary payoff-functions. This can be utilized for numerical schemes: European options with arbitrary payoffs can be priced by implementing numerical integration schemes like high order Gaussian integration. If one is interested in pricing Bermudan options, one can construct a grid at time t [Bermudan day k], and obtain the option values there by making a time step until t [Bermudan day k+1] and using the same representation as above. At the future time step, typically the Gauss integration points will be no grid points, hence interpolation techniques should be applied.



Picture 1:Value of a chooser option on an equity paying discrete dividends. Expiry of the chooser option is in the front. Results obtained by Adaptive Integration, as implemented in the UnRisk PRICING ENGINE.

Similar techniques are available for one factor interest rate models. The differential equation here looks for example [Wilmott] like this:

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+(u-\lambda w)\frac{\partial V}{\partial r}-rV=0$$

The Green Function here looks quite complicated [Shreve, pp. 300-301], because the underlying r is also used for discounting (which is different to Black-Scholes or Black 76). This leads to double integrals, which can be reduced to single integrals in the case of, say, a one-factor Hull-White model.

Upwind Techniques and Streamline Diffusion

The partial differential equations obtained by using (one or more-factor) short rate models or by modelling convertible bonds considering stochastic equity and stochastic interest rates can be interpreted as convection-diffusion-reaction equations. This type of equation is typically found in applications in continuum mechanics. The typical shape of a two factor interest rate model is

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+\rho wq\frac{\partial^2 V}{\partial r\partial l}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial l^2}+(u-\lambda_r w)\frac{\partial V}{\partial r}+(u-\lambda_l w)\frac{\partial V}{\partial l}-rV=0$$

where the coefficient functions are determined by the special interest rate model.

The numerical solution using standard discretisation methods cause severe problems, resulting in high oscillations in the computed values. It is the drift term which is mainly responsible for these difficulties and which forces us to use specifically developed methods for the numerical solution. These methods have to use so-called upwind strategies, in order to obtain stability meaning, very roughly speaking, that you use information at those points where this information comes from ("you follow the direction of the flow"). In trinomial tree methods, it is the up-branching and down-branching which takes into account the upwinding and leads to nonnegative weights which correspond to stability.

The standard streamline diffusion method, which was introduced by Hughes and Brooks for the numerical solution of convection dominated convection-diffusion equations, is such a method and is based on the finite element method. It achieves stability by adding artificial diffusion into the direction of the streamlines, which are mainly determined by the drift. In addition to its good global stability properties it is a method of higher order of convergence, which yields additional advantages compared for example to simple upwinding with finite differences.
The picture shows the value of an option on a zero coupon bond as a function of the short rate r and of the second state variable u of the Hull-White model like in [Rebonato]. Note that this picture should demonstrate the stability and robustness properties of the streamline diffusion method.

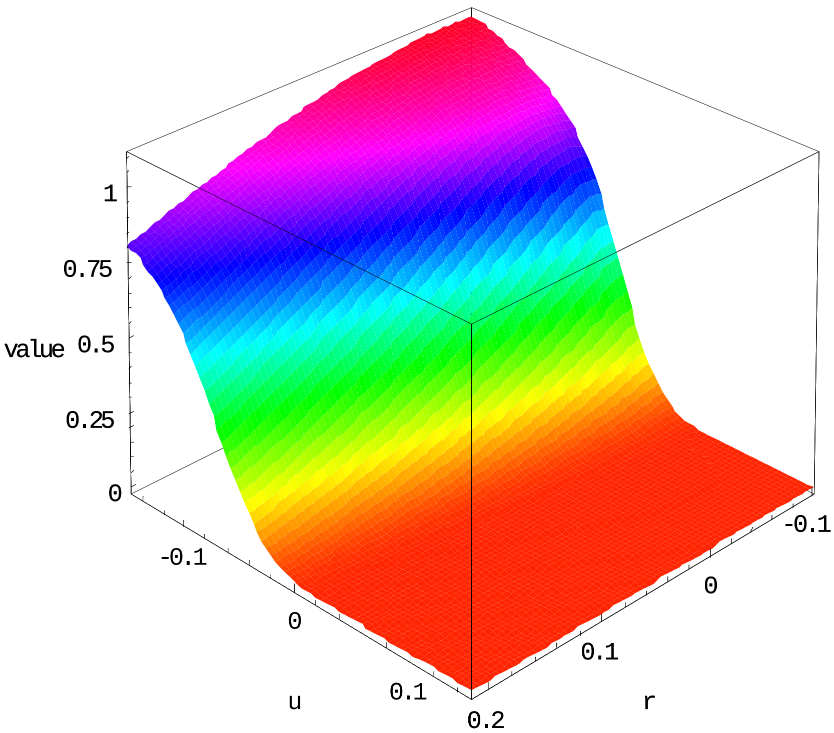


Image source picture 2: MathConsult GmbH, Linz

Inverse Problems and Model Calibration

For the pricing of financial derivatives, the user has to provide input data which describe the random behaviour of the underlying process. Typically the volatility is the most critical input. The standard routine in pricing complex structures is, first, to identify the model parameters from market prices of liquid and actively traded instruments, and second, use the obtained parameters in advanced pricing schemes like the ones described above.

Calibrating model parameters in our PDE framework means the identification of parameters in parabolic differential equations, a problem which is ill-posed in the sense of Hadamard. This means:

a) For given market data, a solution (model parameters) need not exist, or
b) If it exists, the solution need not be unique, or
c) The solution need not depend continuously on the data, meaning that arbitrarily small perturbations of the data might lead to arbitrarily large perturbations of the solution.

For the robust solution of ill-posed problems, so called regularization techniques have to be applied to obtain stable and robust algorithms.

Consider the problem of determining a local volatility function (in the sense of Dupire) $\sigma(S)$ from prices of options with one maturity but with different strike prices. As long as there is no noise in the data, output least square approaches work quite well as the following picture demonstrates

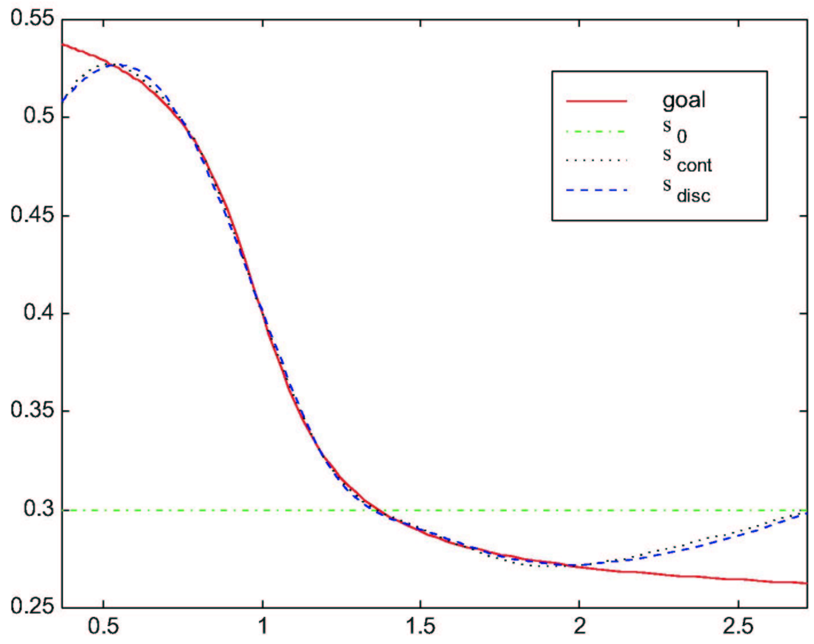
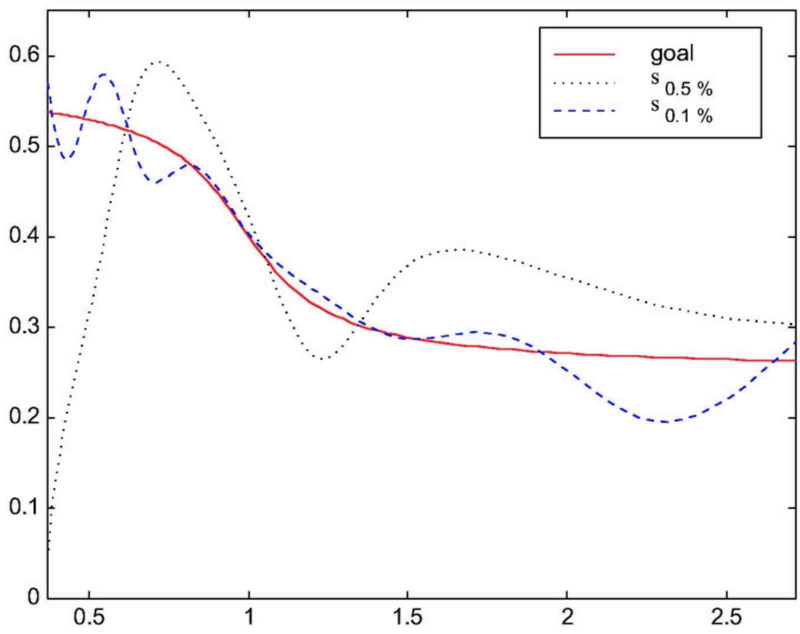
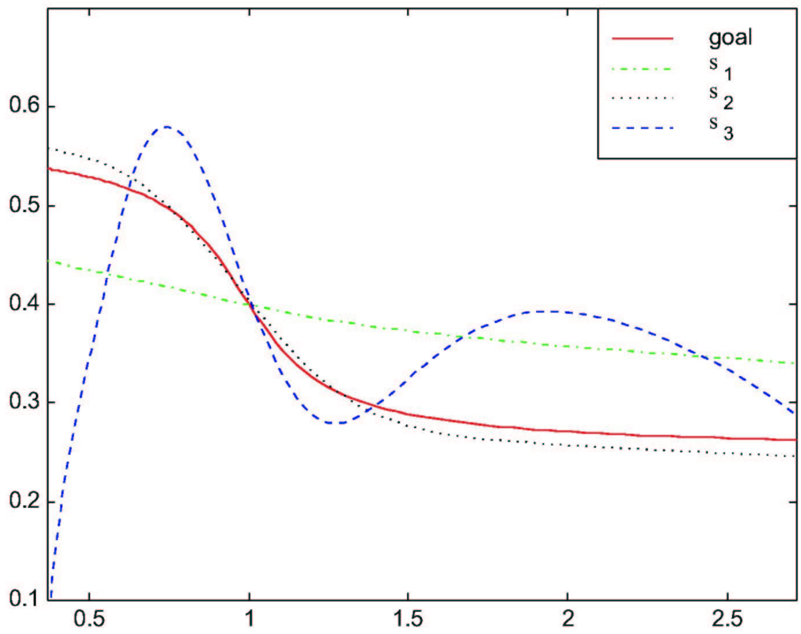


Image Source pictures 3,4,5: Industrial Mathematics Institute, University of Linz

The red line is the (unknown) volatility function, and the spot price of the equity is assumed to be 1. As expected, there is good identification as long as we are not too deep in the money or to deep out of the money. At the extreme ends, the option prices do not contain much information and therefore the solution identified (blue) depends on the starting level (green). If you add noise to the input data (and there is always noise in option prices due to bid-offer spreads), the situation becomes nasty without regularization.



With (quite small) noise levels of 0.1 and 0.5 percent, output least squares techniques lead to oscillating results. But regularization helps:



The dotted curve shows the result for a good choice of the regularisation parameter, the blue curve is obtained from under-regularisation, the green one for over-regularisation. A posteriori techniques (which do not need knowledge on the true solution) for choosing optimal regularization strategies are available.

Resume

Advanced numerical techniques which take into account accuracy, speed, stability and robustness are a must in modern financial engineering. Surprisingly (or not so surprisingly?), algorithms from engineering applications like computational multiphysics problems can and should be applied to computational finance problems.

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